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# Scaling model of a rainfall intensity-duration-frequency relationship

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## Abstract:

Intensity-duration-frequency (IDF) relationships are currently constructed based on an at-site frequency analysis of rainfall data separately for different durations. These relationships are not accurate and reliable since they depend on many assumptions such as distribution selection for each duration; they require a large number of parameters, and are not time-independent.

In this study, scaling properties of extreme rainfall are examined to establish scaling behaviour of statistical non-central moments over different durations. A scale invariance concept is explored for disaggregation (or downscaling) of rainfall intensity from low to high resolution and is applied to the derivation of scaling IDF curves. These curves are developed for gauged sites based on scaling of the generalized extreme value (GEV) and Gumbel probability distributions.

Numerical analysis was performed on annual maximum rainfall series for the province of Ontario, for storm durations of 5, 10, 15, and 30 min (the typical time of concentration for small urban catchments) and 1, 2, 6, 12, and 24 h (the typical time of concentration for larger rural watersheds).

Results show that rainfall does follow a simple scaling process. Estimates found from the scaling procedure are comparable to estimates obtained from traditional techniques; however, the scaled approach was more efficient and gives more accurate estimates compared with the observed rainfall total at all stations. 

KEY WORDS scaling; annual maximum rainfall; intensity-duration-frequency relationships; generalized extreme value (GEV) distribution

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## INTRODUCTION

Information on extreme rainfall characteristics is required in the hydrologic design of structures that control storm run-off. Such information is often expressed as a relationship between intensity-duration-frequency (IDF). The IDF is constructed by performing statistical analysis on annual maxima series (AM) or partial duration series (PDS) by fitting probability distributions for several pre-selected rainfall durations. Such a procedure has several disadvantages including fitting distribution and parameters estimation for each duration, and requirements to extrapolate results to different timescales. The usefulness and accuracy of such procedures is limited because of fitting uncertainties and lack of ability to adequately describe rainfall properties at different timescales (Van Nguyen and Wang, 1996). It is therefore advantageous to develop a model, which could adequately describe rainfall characteristics through a continuum of timescales including inferences at time resolutions that may not have been observed.

The purpose of this study is to investigate 'scale invariance' or 'scaling' properties of rainfall for derivation of IDF relationships.

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## DATA

Numerical analysis was performed on annual maximum rainfall series from the province of Ontario for storm durations of 5, 10, 15, and 30 min (the typical time of concentration for small urban catchments) and 1, 2, 6, 12, and 24 h (the typical time of concentration for larger rural watersheds). The data was obtained from the Atmospheric Environmental Services (AES) Branch of Environmental Canada. A total of five stations from Eastern Ontario were chosen for the analysis and are shown in Table I. The five stations were selected from Eastern Ontario, based on record length, and availability of current datasets. The record lengths ranged from 24 to 38 years and data sets ranged from 1957 to 1998, respectively. Record lengths vary per storm duration for a given site. For Cornwall, Kingston, Ottawa CDA, and Ottawa International Airport, the record length of 34, 38, 38, and 32 is consistent for storm durations ranging from 5 min to 12 h. However, the record length for the 24-h rainfall is different than for other storm durations, with values of 31, 30, 30, and 24 for Cornwall, Kingston, Ottawa CDA, and Ottawa International Airport, respectively. Three different record lengths exist for Kemptville storm durations: 26 years (5–15 min); 27 years (30 min–12 h) and 24 years (24-h storm).

## METHODOLOGY

The space–time structure and variability of rainfall is very complex and despite much research effort in dynamic (numerical) modelling and coupling of dynamic and statistical descriptions of rainfall, accurate and practical results have not been achieved. To solve the physical equations, many parameters and simplifying assumptions are required making these models too complicated for practical applications and they are not consistent at different timescales. On the other hand, recently developed scaling concepts (Gupta, 2004) offer a new and better possibility to represent rainfall over different scales based on the empirical evidence that rainfall exhibits scale invariance symmetry. Scale invariance symmetry implies that the statistical properties of rainfall at different scales are related to each other by a scale-changing operator involving only the scale ratio. Scale models of rain have evolved from fractal geometry, mono-fractal fields, multi-fractals, generalized scale invariant models, and universal multi-fractals (Foufoula-Georgiou and Krajewski, 1995). Unfortunately, these models were found to be insufficient to describe all features of rainfall fields, such as anisotropy and stratification. Recently, Burlando and Rosso (1996) provided a practical scaling model that can be used to derive depth-duration-frequency relationships using lognormal probability distribution. In this study, scaling properties of extreme rainfall are examined based on the scaling behaviour of the statistical non-central moments (NCM) over different durations using a simple scaling approach (Van Nguyen, 2000). The approach consists of examining the scale invariance properties of extreme rainfall time series based on the scaling

Table I. Stations used in the analysis

| Station | ID | Record length | | Time period | Latitude | Longitude |
|---|---|---|---|---|---|---|
| | | 5 min–12 h | 24 h | | | |
| Cornwall | 6101901 | 34 | 31 | 1957–1993 | 45°02′ | 74°48′ |
| Kemptville | 6104025 | 26[a], 27[b] | 21 | 1970–1996 | 45°00′ | 75°38′ |
| Kingston | 6104175 | 38 | 30 | 1961–1998 | 44°14′ | 76°29′ |
| Ottawa CDA | 6105976 | 38 | 30 | 1959–1998 | 45°23′ | 75°43′ |
| Ottawa Int'l A | 6106000 | 32 | 24 | 1967–1998 | 45°19′ | 75°40′ |

[a] For storm durations: 5, 10, and 15 min.
[b] For storm durations: 30 and 60 min, and 1, 2, 6, and 12 h.

behaviour of the statistical NCM over different durations employing the generalized extreme value (GEV) distribution and L-moments technique of parameter estimation.

There is no general agreement about which probability distribution should be used for the frequency analysis of extreme rainfalls. The GEV distribution and its special form, the Gumbel distribution, are the most commonly used (Adamowski *et al.*, 1996) to model the AMS data. Many methods can be used to estimate parameters of the selected distribution, including the method of L-moments that is particularly suited for small sample sizes (Hosking, 1990).

*GEV*

Random hydrological variables that are extremes, such as maximum rainfall and floods, are often described by several extreme value (EV) distributions developed by Gumbel (1954). The GEV distribution's cumulative distribution function (cdf) can be written as

$$F(x) = \exp\left\{-\left[1 - \frac{k(x-\xi)}{a}\right]^{1/k}\right\} \quad \text{for} \quad k \neq 0 \tag{1}$$

The Gumbel distribution results when the $k$ variable in Equation (1) is zero. For $k < 0$, the Weibull or EV type II distribution is derived, and the GEV type II distribution results when $k > 0$.

The parameters of the GEV distribution in terms of L-moments are given by Maidment (1993)

$$k \approx 7{\cdot}817740c + 2{\cdot}930462c^2 + 13{\cdot}641492c^3 + 17{\cdot}206675c^4 \tag{2}$$

$$\alpha = \frac{k\lambda_2}{\Gamma(1+k)(1-2^{-k})} \tag{3}$$

$$\xi = \lambda_1 + \frac{\alpha}{k[\Gamma(1+k)-1]} \tag{4}$$

where the variable $c$ is defined by

$$c = \frac{2\beta_1 - \beta_0}{3\beta_2 - \beta_0} - \frac{\ln(2)}{\ln(3)} \tag{5}$$

where $\lambda_1, \lambda_2$, and $\lambda_3$ are L-moments of order 1, 2, and 3, respectively; and $\beta_0$ and $\beta_1$ are the probability weighted moments (PWM).

A good approximation for the gamma distribution, when $k$ ranges between 0 and 1, is shown by the following equation

$$\Gamma(1+k) = 1 + \sum_{i=1}^{5} a_1 k^i + \varepsilon \tag{6}$$

where $a_1 = -0{\cdot}5\ 748\ 646$, $a_2 = 0{\cdot}9\ 512\ 363$, $a_3 = -0{\cdot}6\ 998\ 588$, $a_4 = 0{\cdot}4\ 245\ 549$, $a_5 = -0{\cdot}1\ 010\ 678$, $|\varepsilon| = 5 \times 10^{-5}$.

The quantiles for the GEV distribution can be computed from

$$x_p = \xi + \frac{\alpha}{k}\{1 - [-\ln(p)]^k\} \tag{7}$$

where $p$ is the desired cumulative probability. In this paper, the GEV distribution was used as a parent distribution for the scaling procedure.

*Gumbel*

IDF curves are often fitted with the extreme value type I (EVI) distribution developed by Gumbel (1954) and it is still the most often used distribution by many national meteorological services in the world to describe



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rainfall events (World Meteorological Organization, 1981). It will also be used in this study along with the method of moments.

The EVI distribution requires the first two moments, the mean and standard deviation, to be computed from the data. The skewness is fixed in the EVI with a constant value of 1·14. The method of moment estimates for the parameters are (Maidment, 1993)

$$\hat{\alpha} = 0{\cdot}7797s \tag{8}$$

$$\xi = \overline{x} - 0{\cdot}45s \tag{9}$$

where $\overline{x}$ and $s$ denote the mean and standard deviation of the observed data set.

The quantiles for the Gumbel distribution can be computed from

$$x_p = \xi - \alpha \ln[-\ln(p)] \tag{10}$$

IDF estimates were computed using the Gumbel distribution, because this method is currently used in practice. Estimates computed by scaling, GEV, and the Gumbel distribution were compared to the observed data for all stations.

*A scaling generalized extreme value distribution*

The basic theoretical development of scaling has been investigated by many authors including Gupta and Waymire (1990); Van Nguyen (2000). The relationship between NCM of order $k$, $\mu_k$, and the variable $x$ can be written as

$$\mu_p = \alpha(k)x^{\beta k} \tag{11}$$

where $\alpha(k) = \mathrm{E}\{\mathrm{f}^k(1)\}$ and the scaling exponent $B(k) = Bk$. If the exponent $B(k)$ is not a linear function, then the scaling process is identified as being multi-scaling (Gupta and Waymire, 1990). The simple scaling assumption is valid if a log–log plot of the NCM versus duration $x$ is linear.

Van Nguyen (2000) showed that the $k$-th order NCM of the GEV distribution is given as follows:

$$\mu_k = \left(\xi + \frac{\alpha}{\kappa}\right)^k + (-1)^k \left(\frac{\alpha}{\kappa}\right) \Gamma(1 + kx) + k \sum_{i=1}^{k-1} (-1)^i \left(\frac{\alpha}{\kappa}\right)^i \left(\xi + \frac{\alpha}{\kappa}\right)^{k-1} \Gamma(1 + ik) \tag{12}$$

where $\Gamma()$ is the gamma function.

The GEV distribution was used in this study as the parent distribution for the scaling procedure. To relate the GEV distribution at time $t$ to the scaled distribution at time $\lambda t$, the following equation was used (Van Nguyen, 2000)

$$x_p(\lambda t) = \lambda^B X_p(t) \tag{13}$$

where $X_p(t)$ is the estimate found by using the GEV distribution, defined in Equations (7) and (10) and $\lambda^B$ is defined by

$$\lambda^B = \frac{\mu_1(\lambda t)}{\mu_1(t)} \tag{14}$$

where $\mu_1(\lambda t)$ and $\mu_1(t)$ are the first order NCM for rainfall of storm duration $\lambda t$ and $t$, respectively. In this paper, the GEV distribution was used to determine the rainfall intensities at a storm duration of 1 h. Then Equations (13) and (14) were used to scale the 1 h distribution to a 5, 10, 15, and 30 min and 2, 6, 12, and 24 duration, respectively.



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*Model performance*

As an indication of goodness of fit between the observed and predicted values the coefficient of determination ($R^2$), the root mean square error (RMSE), and the relative root mean square error (RRMSE) were calculated. The RMSE is defined by

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_{\text{obs}} - X_{\text{est}})^2} \tag{15}$$

where $X_{\text{obs}}$ is the observed rainfall intensity, and $X_{\text{est}}$ is the estimated rainfall intensity found using the GEV, Gumbel, and scaled distributions. The RRMSE is similarly calculated using

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\frac{(X_{\text{obs}} - X_{\text{est}})^2}{\mu_{\text{obs}}}} \tag{16}$$

where $\mu_{\text{obs}}$ is the mean of the observed rainfall data.

## RESULTS AND DISCUSSION

*Scale invariance properties of rainfall*

The scaling properties of rainfall data were investigated by computing the first three NCMs for each duration, and then by examining the log–log plots of the NCMs against their duration. Table II shows the scaling exponents for the storm intervals and corresponding $R^2$ values for all five stations used in the analysis. Figures 1 and 2 illustrate the relationship between NCMs versus short and long duration storms and a plot of the scaling exponents versus the order of moments for the Ottawa International Airport. The values from Table II were found by using the graphs depicted for the Ottawa International Airport.

The difference in the degree of steepness in the slopes for the short (5 min–1 h) and long (1–24 h) duration storms indicate that two different scaling regimes exist for rainfall. This can be observed by a steeper slope found in short duration storms compared to long duration storms. The plots indicate that the relationships between NCMs and durations are linear having two different slopes with a breaking point at the 1-h duration. This property perhaps suggests an existence of two different regimes with a transition in storm dynamics from high variability convective storms (less than 1-h duration) to a smaller variability of frontal storms of longer duration than 1 h.

Table II. Scaling exponents and $R^2$ values for two storm intervals

| Station | Interval | Order | | | $R^2$ |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | |
| Ottawa | 5 min–1 h | 0·3984 | 0·79 | 1·1728 | 0·9997 |
| | 1–24 h | 0·2159 | 0·4184 | 0·6136 | 0·9981 |
| Ottawa CDA | 5 min–1 h | 0·4046 | 0·8264 | 1·263 | 0·999 |
| | 1–24 h | 0·2059 | 0·4034 | 0·5944 | 0·999 |
| Kingston | 5 min–1 h | 0·3947 | 0·79 | 1·1838 | 1 |
| | 1–24 h | 0·2686 | 0·5342 | 0·7979 | 0·9999 |
| Kemptville | 5 min–1 h | 0·3284 | 0·662 | 1·0114 | 0·9994 |
| | 1–24 h | 0·2339 | 0·4371 | 0·617 | 0·9863 |
| Cornwall | 5 min–1 h | 0·3969 | 0·7703 | 1·1298 | 0·9982 |
| | 1–24 h | 0·2616 | 0·5156 | 0·7612 | 0·9993 |



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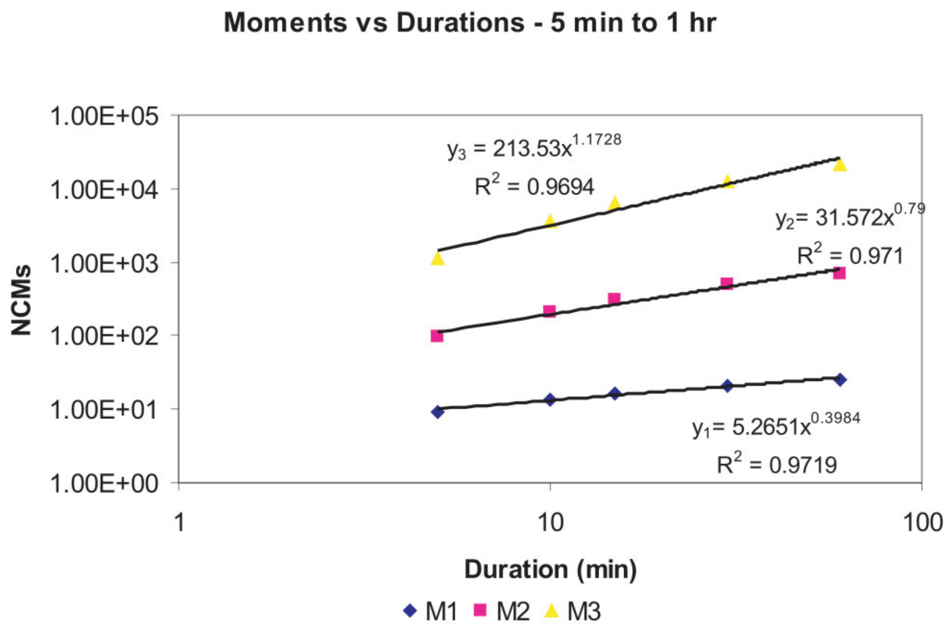
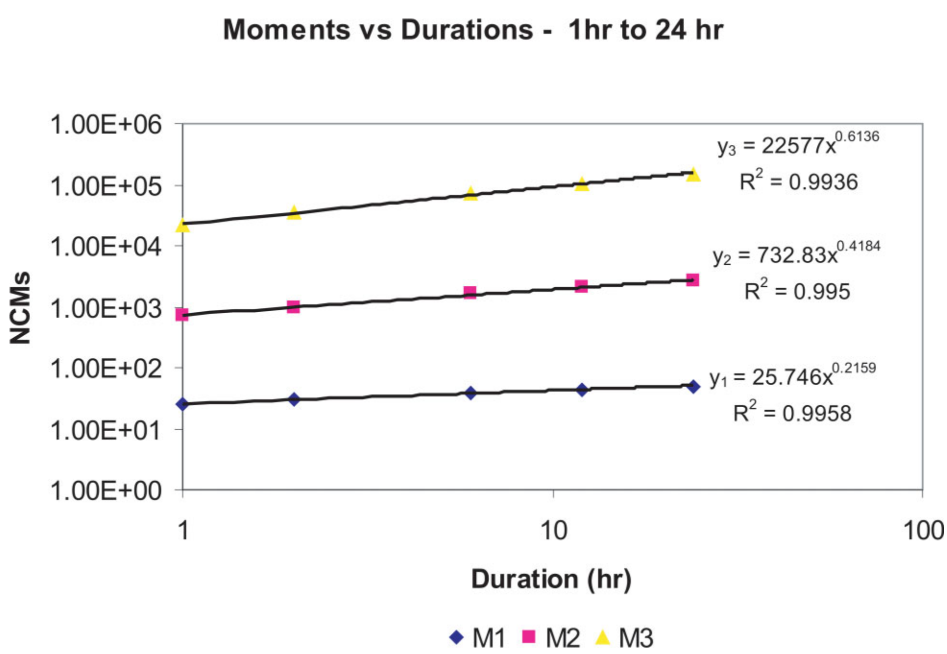



(a) Short duration storms




(b) Long duration storms

Figure 1. Log–log plot of NCM's versus duration for the Ottawa Airport. (a) short duration storms, (b) long duration storms

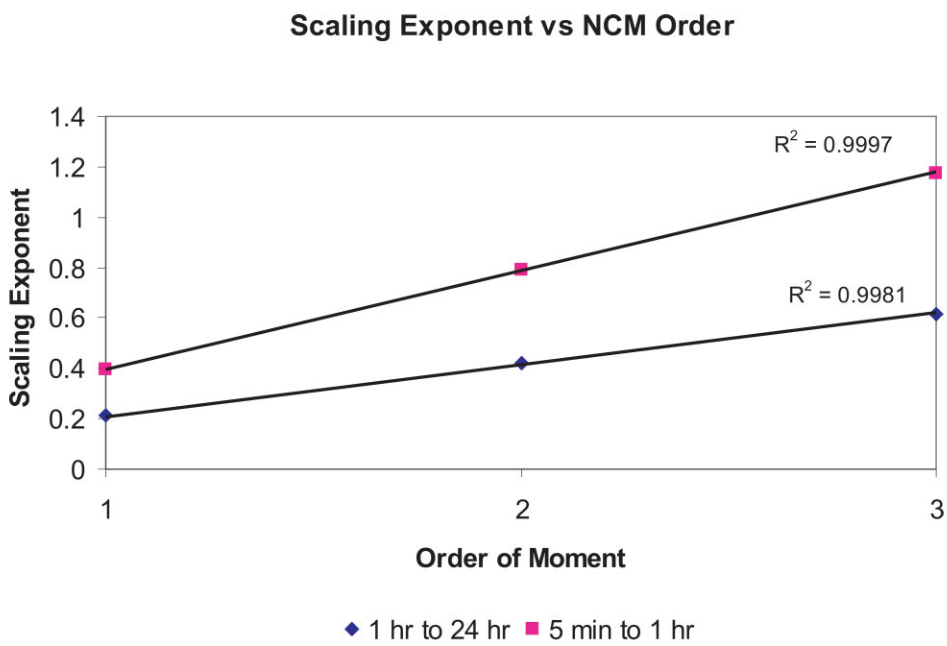



Figure 2. Scaling exponent versus NCM for the Ottawa Airport

The linearity in slope found in the NCM versus log-durations plots illustrates that rainfall follows a simple scaling process. If the exponent is not a linear function of NCM's, then rainfall would follow a multi-scaling process. The slopes of the NCM's versus log-duration plots define the $B(k)$ coefficient shown in Equation (11), and listed in Table II. The high correlation coefficients for each duration interval range from 0·986 to 1 indicating a strong validity of the simple scaling property of extreme rainfall.

*Estimation of annual extreme rainfall*

Figure 3 illustrates the scaled annual extreme (AE) and observed AE rainfall versus probability for the Ottawa Airport site. Figure 3(a) indicates that the scaled AE estimates are similar to the observed, especially for storm durations less than 1 h. This result was typical for all stations analysed in this paper. Figure 3(b) displays results for a 24-h storm. The scaling procedure does well to predict the observed series; however, it appears that the scaling process has problems predicting the high outlier value, which might exist at some rainfall stations. Typically, the correlation between scaled and observed data for storm durations larger than 1 h was weaker than for short duration storms.

Figure 4 illustrates the plots of the scaled (estimated extreme rainfall) versus observed rainfall performed for the Ottawa CDA site and Table III shows the correlation coefficients for all durations and sites. For three sites (Ottawa CDA, Kemptville, and Cornwall), the $R^2$ coefficients ranged from 0·9 to 0·98, indicating similarity between the series. The other two sites also yielded similar results for short duration storms, but the accuracy to predict long duration storms decreased.

Table IV gives cross correlation analysis results (Adamowski and Bougadis, 2003) for all storm durations for the St Lawrence region, which encompasses all sites analysed in this study. As an initial indicator of spatial correlation, cross correlation coefficients were computed. The regions were then tested for homogeneity (H) using the L-moments procedure developed by Hosking and Wallis (1997). High cross correlation coefficients are clearly evident for storm durations greater than 2 h, with coefficients ranging from 0·1 to 0·275. The heterogeneity measures are highly negative, ranging from −1 to −1·76. Hosking and Wallis (1997) states that negative H values can occur, indicating that there is less dispersion among at-site sample



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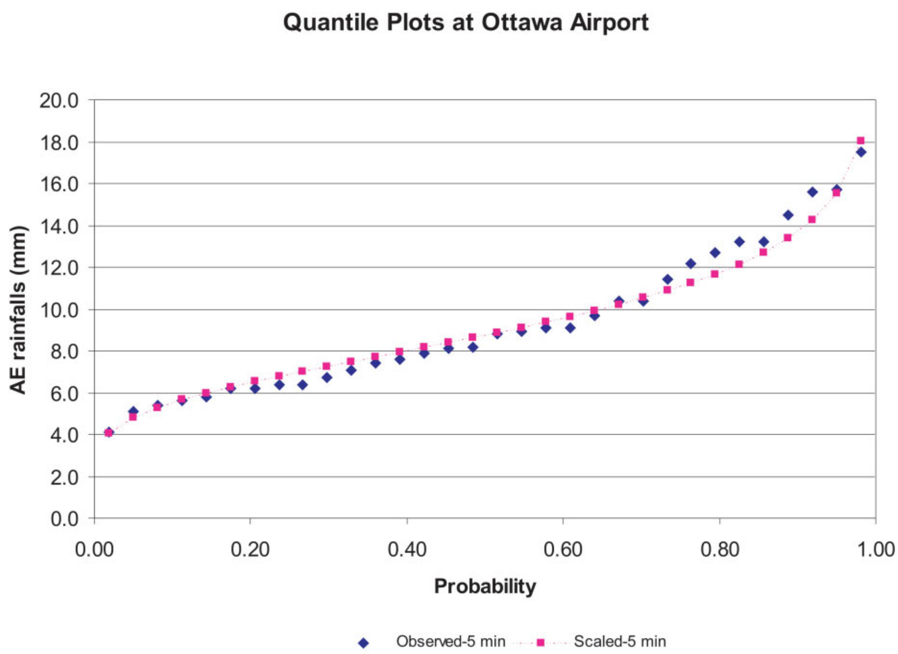
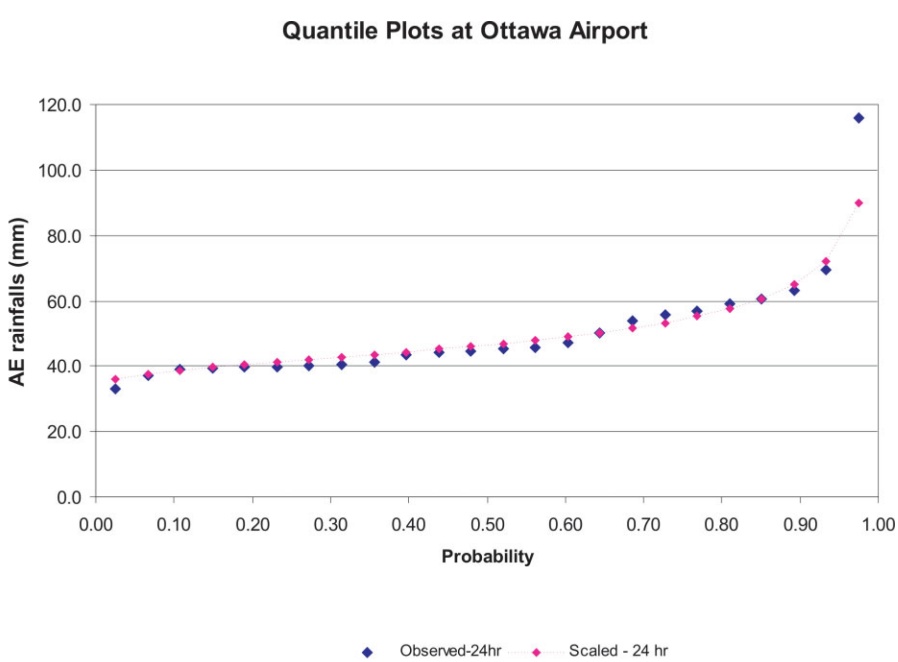



(a) Five minute storm




(b) Twenty-four hour storm

Figure 3. Quantile plots for Ottawa Airport. (a) 5-min storm, (b) 24-h storm

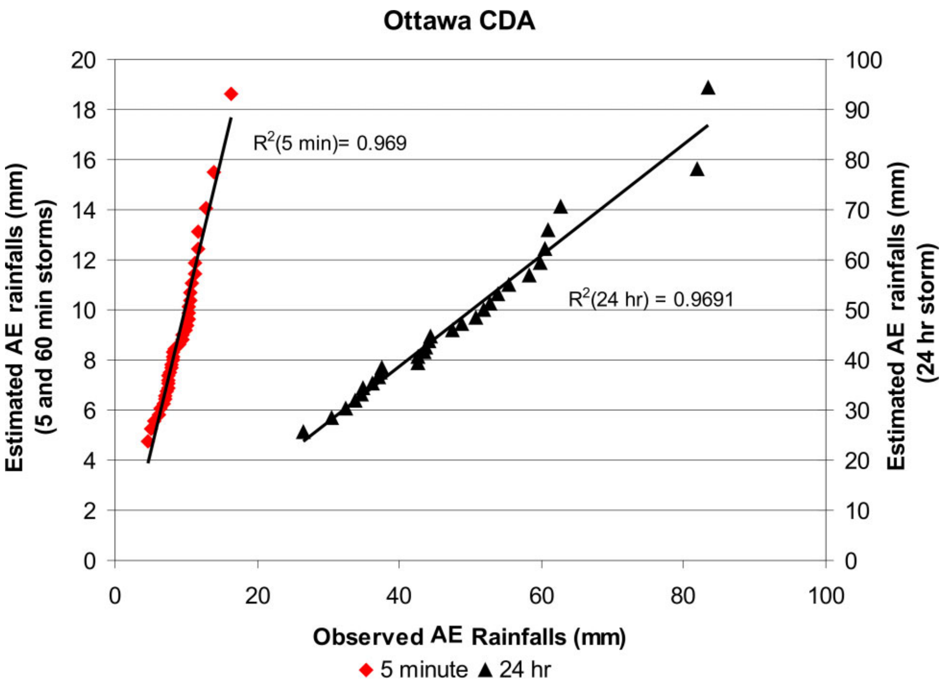



Figure 4. Estimated AE versus observed AE for the 5-min and 24-h storms for Ottawa CDA

Table III. Correlation coefficients ($R^2$) for all durations and sites

| Duration | $R^2$ | | | | |
|---|---|---|---|---|---|
| | Ottawa Airport | Ottawa CDA | Kingston | Kemptville | Cornwall |
| 5 | 0·955 | 0·969 | 0·972 | 0·919 | 0·97 |
| 10 | 0·978 | 0·937 | 0·972 | 0·896 | 0·972 |
| 15 | 0·949 | 0·952 | 0·919 | 0·91 | 0·948 |
| 30 | 0·959 | 0·965 | 0·968 | 0·956 | 0·977 |
| 2 | 0·915 | 0·972 | 0·981 | 0·965 | 0·952 |
| 6 | 0·674 | 0·983 | 0·811 | 0·97 | 0·967 |
| 12 | 0·745 | 0·975 | 0·755 | 0·981 | 0·902 |
| 24 | 0·749 | 0·969 | 0·825 | 0·933 | 0·922 |

L-CV values than what would be expected from a homogeneous region. Hosking and Wallis also indicated that the most likely cause of H being negative is high positive correlation between the data values at different sites in a proposed region. The results for storm durations less than 1 h are not as conclusive, with H values ranging from −0·2 to 1, respectively, indicating that the storm durations are weakly correlated in space. It can therefore be concluded that the data comes from a homogeneous region, restricting the results of this study to this region. Further study is needed to examine whether the results would be applicable to other regions.

Model performance variables (RMSE and RRMSE), and the best distribution, which most accurately fits the observed data were performed for all sites and storm durations. As already mentioned, the scaling estimates for short duration storms had a higher degree of accuracy than long duration storms. The Gumbel distribution was the best distribution to select for the two Ottawa sites; however, the GEV distribution was the best distribution to select in Cornwall. The selection of distribution for Kingston and Kemptville was not obvious;



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Table IV. Spatial correlation (after Adamowski and Bougadis, 2003)

| Duration | Cross correlation | Heterogeneity measure |
|---|---|---|
| 5 min | 0·082 | −0·18 |
| 10 min | 0·023 | −0·72 |
| 15 min | 0·005 | −0·77 |
| 30 min | 0·000 | −0·5 |
| 1 h | 0·049 | −1·04 |
| 2 h | 0·106 | −1·76 |
| 6 h | 0·211 | −1·12 |
| 12 h | 0·275 | −1·07 |

however, it is clear that assuming the Gumbel distribution as the best distribution to be used in rainfall analysis is not entirely valid. The GEV distribution was the best distribution to accurately depict observed data for a number of durations and stations.

*Estimation of IDF rainfall estimates*

The graphical results of IDF estimates are shown in Figure 5 for the Ottawa Airport station. It can be seen that the scaled estimates are relatively close to observed estimates for short duration storms. For long duration storms, a greater discrepancy exists for all stations when the record lengths are greater than 10 years. Similar observations can be made for other stations used in this study.

## CONCLUSIONS

The results of this study show that rainfall follows a simple scaling process with two different scaling regimes: 5 min–1 h and 1–24 h. Results found from scaling estimates are very similar to observed data for short duration and low return periods.

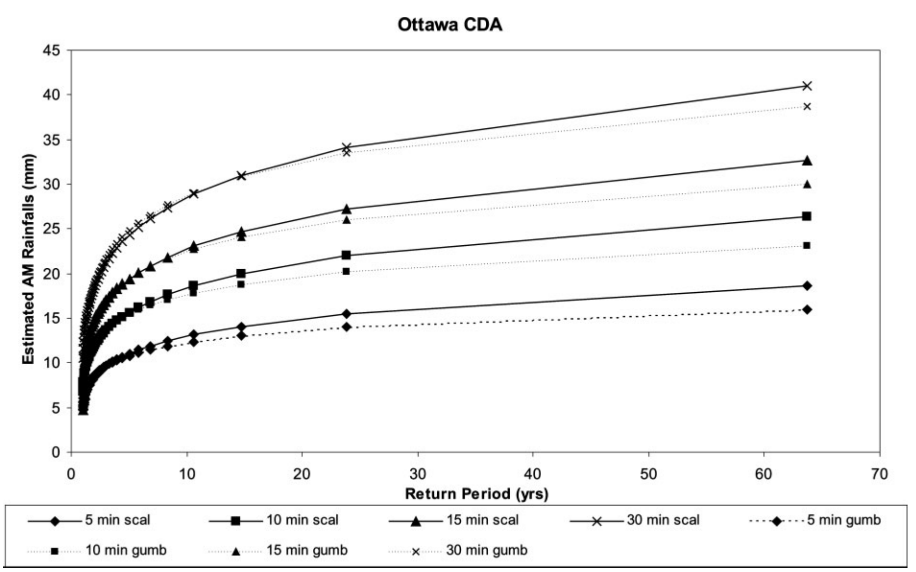


Figure 5. IDF curves for short duration storm for Ottawa CDA



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The benefit of using the principles of scaling is that it reduces the amount of parameters required to compute the quantiles. If data is missing from a station, then the first order moment of the duration in question is the only parameter required to compute the quantiles. If that station belongs in a homogeneous region, then the regional $t$ minute first order moment can be used to determine estimates. In practical applications, short duration storms and return periods less than 10 years are used to size drainage pipes for minor system analysis. This paper shows that the scaling estimates are most accurate for durations less than 1 h and return periods less than 10 years, respectively.

Results of this study are of significant practical importance because statistical rainfall inferences can be made from a higher aggregation model (i.e. observed daily data) to a finer resolution model (i.e. less than 1 h, that might not have been observed). This is important since daily data are more widely available from standard rain gauge measurements, but data for short durations are often not available for the required site. The findings from this study can be further extended for regional analysis.

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