Mechanics & Materials 1

Chapter 15
Stress and Strain Transformation

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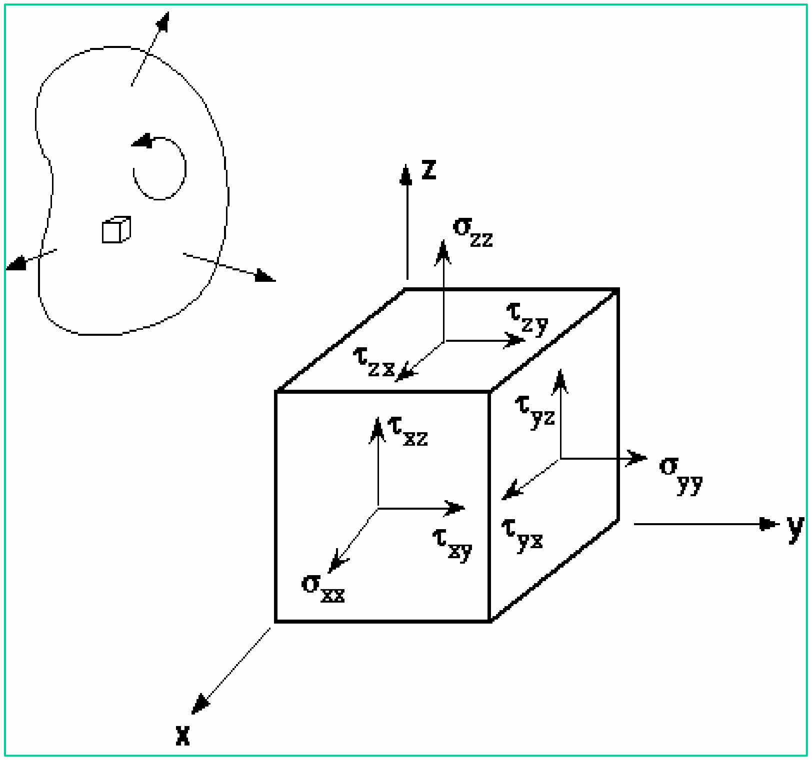
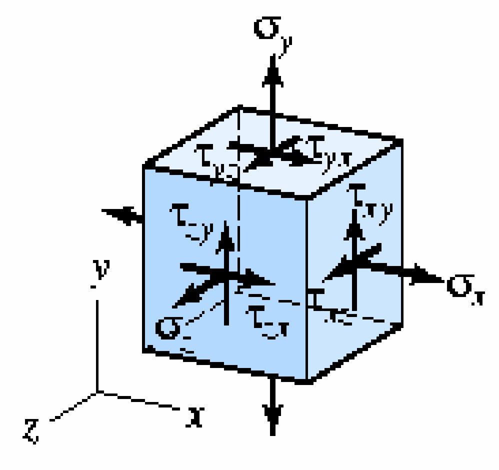
General State of Stress

- In general, the three dimensional state of stress at a point in a body can be represented by nine components:
- $\sigma_{xx} \sigma_{yy}$ and σ_{zz}: Normal stresses
- τ_{xy} τ_{yx} τ_{xz} τ_{zx} τ_{yz} and τ_{zy} : Shear stresses
- By equilibrium, we can show that there are only six independent components of the stress σ_{xx} ,σ_{yy} ,σ_{zz} ,τ_{xy}, τ_{xz}, and τ_{yz}

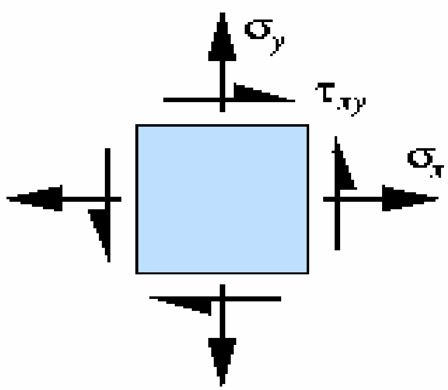


Plane Stress

- In much of engineering stress analysis, the condition of plane stress applies.
- Plane Stress: one of the three normal stresses, usually σ_z vanishes and the other two normal stresses σ_x and σ_y, and the shear stress τ_{xy} are known.



General state of stress.

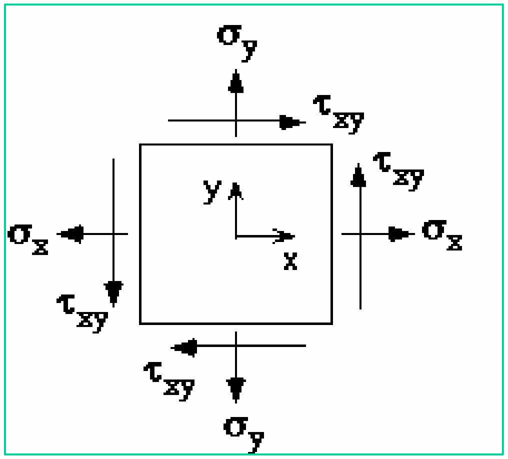


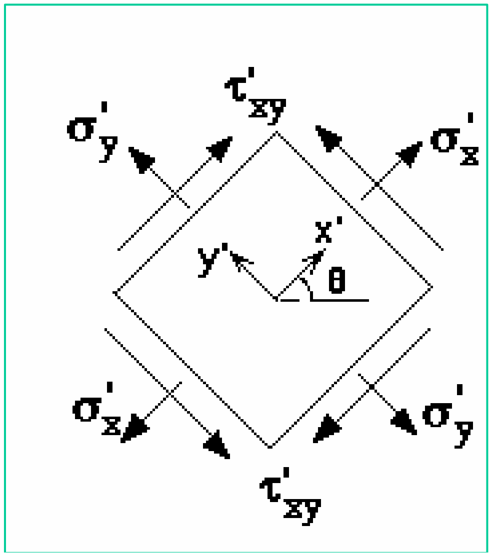
Plane Stress Transformation: Finding Stresses on Various Planes

- General Problem:

- * Given two coordinate systems, x- y and x' - y', and a stress state defined relative to the first coordinate system xyz :σ_x σ_y, τ_{xy}

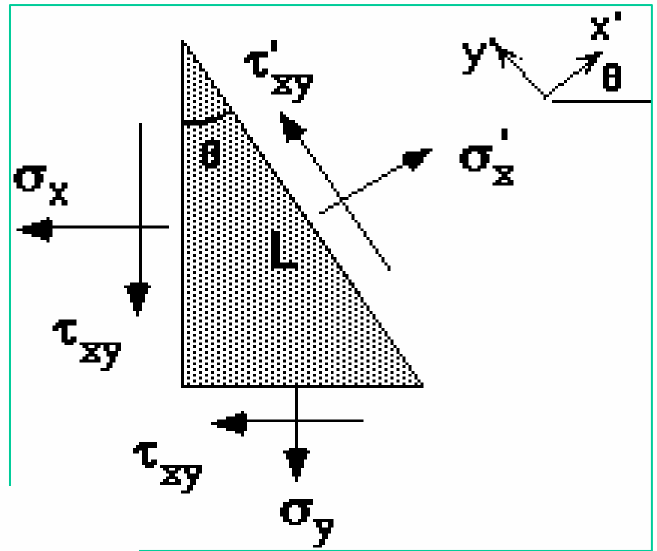
- * Find the stress components relative to the second coordinate system x'y'z' : σ'_x σ'_y, τ'_{xy}





Plane Stress Transformation

- Consider a triangular block of uniform thickness, t:
- For equilibrium:



$$\sum F_{x'} = tL\sigma'_x$$
$$- t(L\cos\theta)\sigma_x \cos\theta - t(L\cos\theta)\tau_{xy}\sin\theta$$
$$- t(L\sin\theta)\sigma_y \sin\theta - t(L\sin\theta)\tau_{xy}\cos\theta = 0$$

$$\sum F_{y'} = tL\tau'_{xy}$$
$$+ t(L\cos\theta)\sigma_x \sin\theta - t(L\cos\theta)\tau_{xy}\cos\theta$$
$$- t(L\sin\theta)\sigma_y \cos\theta - t(L\sin\theta)\tau_{xy}\sin\theta = 0$$

Plane Stress Transformation

- Simplifying:

$$\sigma'_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

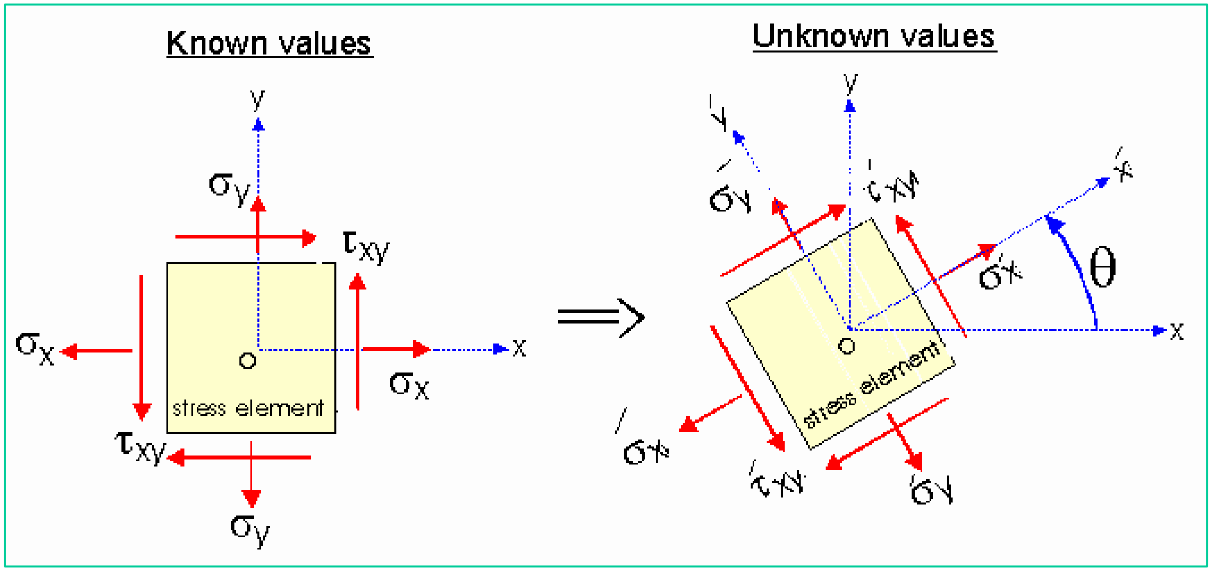
$$\tau'_{xy} = \tau_{xy}(\cos^2\theta - \sin^2\theta) - (\sigma_x - \sigma_y)\sin\theta\cos\theta$$

- Using :

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta) \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

Transformation Equations for Plane Stress



$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\Rightarrow \sigma'_x + \sigma'_y = \sigma_x + \sigma_y$$

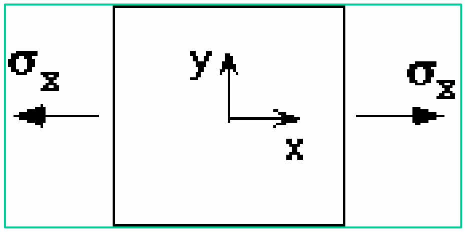
Special Cases of Plane Stress

- 1. Uniaxial Stress State: $\sigma_y = \tau_{xy} = 0$

$$\sigma'_x = \frac{1}{2}\sigma_x(1+\cos 2\theta) = \sigma_x \cos^2\theta$$

$$\sigma'_y = \frac{1}{2}\sigma_x(1-\cos 2\theta) = \sigma_x \sin^2\theta$$

$$\tau'_{xy} = -\frac{1}{2}\sigma_x \sin 2\theta = -\sigma_x \sin\theta\cos\theta$$

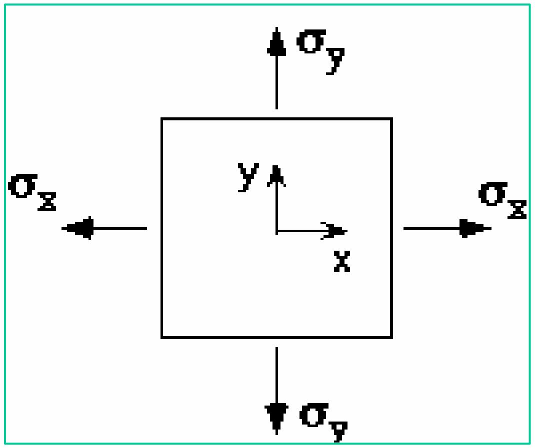


- 2. Biaxial Stress State: $\tau_{xy} = 0$

$$\sigma'_x = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta$$

$$\sigma'_y = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta$$



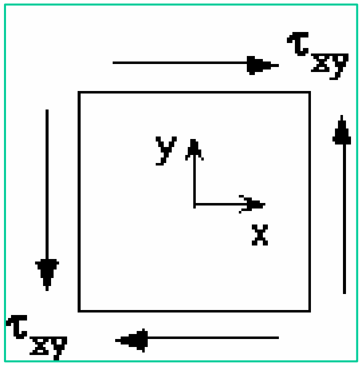
Special Cases of Plane Stress

- 3. Pure Shear: $\sigma_x = \sigma_y = 0$

$$\sigma'_x = \tau_{xy} \sin 2\theta = 2\tau_{xy} \sin\theta \cos\theta$$

$$\sigma'_y = -\tau_{xy} \sin 2\theta = -2\tau_{xy} \sin\theta \cos\theta$$

$$\tau'_{xy} = \tau_{xy} \cos 2\theta = \tau_{xy}(\cos^2\theta - \sin^2\theta)$$



Principal Stress

- σ'_x varies as a function of the angle θ

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

- The maximum and minimum values of σ'_x are called the principal stresses. To find the max and min values: $\frac{d\sigma'_x}{d\theta} = 0$

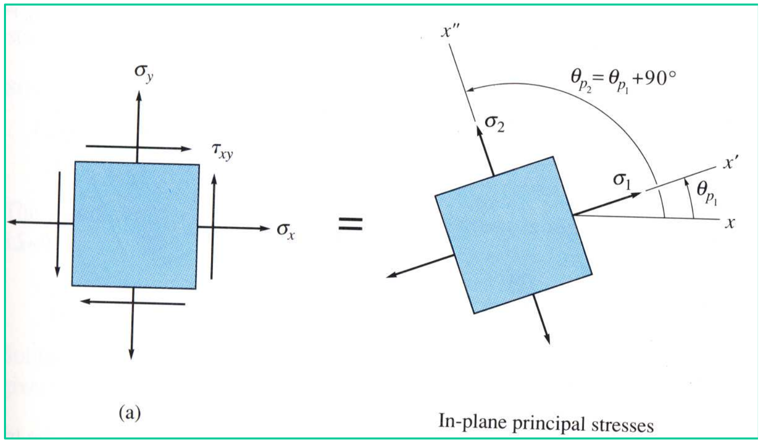
$$\frac{d\sigma'_x}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\Rightarrow \tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Where θp defines the orientation of the principle planes on which the principle stress act.

Principal Stress

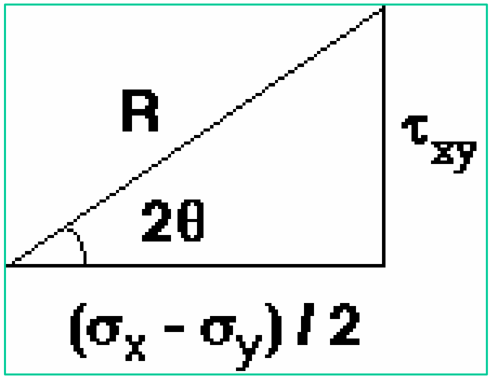
- Two values of $2\theta_p$ in the range of 0 to 360 satisfy this equation.
- These two values differ by 180° so that θ_p has two values that differ by 90°, one between 0 and 90° and the other between 90° and 180°.
- For one of the angles θ_p , the stress is a maximum principal stress (σ_1) and for the other it is a minimum (σ_2).
- Because the two values of θ_p are 90° apart, ==> the principal stress occur on mutually perpendicular planes.



In-plane principal stresses

Principal Stress

- To calculate θ_p consider the triangle



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\cos 2\theta_p = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{R}$$

- Subbing back in yields the principal stresses:

$$\sigma_1 = \sigma'_x(\theta_p)$$

$$\sigma_1 = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \sigma'_x(\theta_p + 90°)$$

$$\sigma_2 = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- OR

$$\Rightarrow \sigma_{1,2} = \sigma_{avg} \pm R$$

where

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

Shear Corresponding to Principal Stress

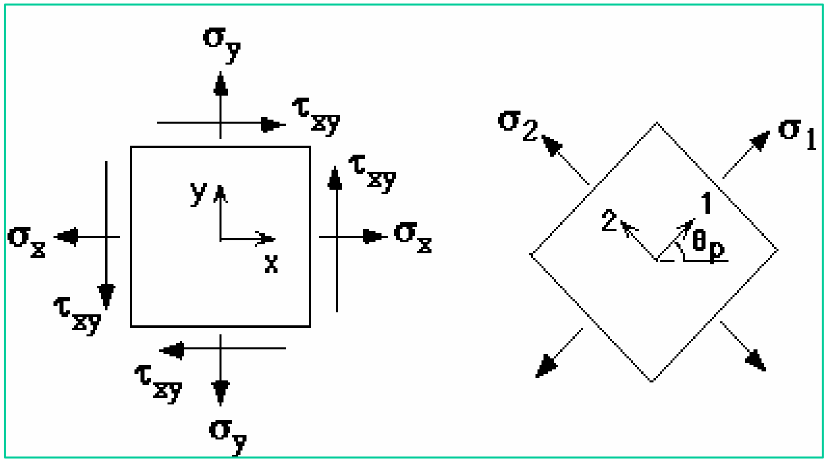
The shear stress corresponding to the principal stress direction is given by:

$$\tau_{12} = \tau'_{xy}(\theta_p)$$

$$= -\left(\frac{\sigma_x - \sigma_y}{2}\right)\frac{\tau_{xy}}{R} + \tau_{xy}\left(\frac{\sigma_x - \sigma_y}{2R}\right)$$

$$= 0$$

The shear stress is identically zero in the principal stress directions! (biaxial stress state)



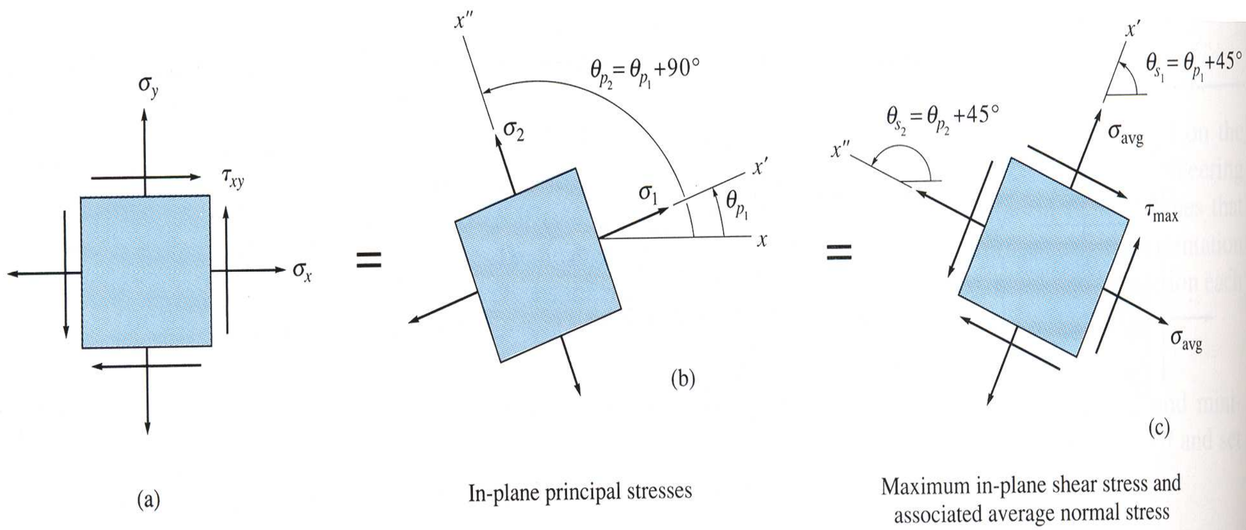
Maximum Shear Stress

- To find maximum shear:

$$\frac{d\tau'_{xy}}{d\theta} = 0$$

$$\frac{d\tau'_{xy}}{dx} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \quad \Rightarrow \tan 2\theta_s = -\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)$$

- Where θ_s defines the angle of the planes of maximum shear stress.



Maximum Shear Stress

- From trigonometry

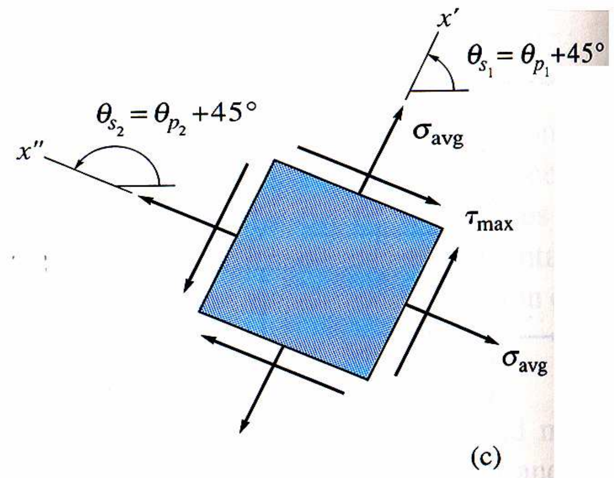
$$\theta_s = \theta_p \pm 45°$$

- The planes of maximum shear stress occur at 45° to the principal planes.
- Subbing back in yields max shear:

$$\tau_{max} = \tau'_{xy}(\theta_s)$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = R$$

$$= \frac{\sigma_1 - \sigma_2}{2}$$



Maximum in-plane shear stress and associated average normal stress

Maximum Shear Stress

- The normal stress corresponding to the max shear stress direction is given by:

$$\sigma'_x = \sigma'_x(\theta_s)$$

$$= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\frac{\tau_{xy}}{R} - \tau_{xy}\left(\frac{\sigma_x - \sigma_y}{2R}\right)$$

$$= \frac{\sigma_x + \sigma_y}{2} = \sigma_{avg}$$

Summary of Equations

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \theta_s = \theta_p \pm 45^\circ$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

$$\sigma_1 = \sigma_{avg} + R$$

$$\sigma_2 = \sigma_{avg} - R$$

$$\tau_{\max} = R = \frac{\sigma_1 - \sigma_2}{2}$$

Mohr's Circle for Plane Stress

- Recall the plane stress transformation equations:

$$\sigma'_x = \sigma_{avg} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

- Rearrange to get

$$\left(\sigma'_x - \sigma_{avg}\right)^2 + \left(\tau'_{xy}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

$$\Rightarrow \left(\sigma'_x - \sigma_{avg}\right)^2 + \left(\tau'_{xy}\right)^2 = R^2$$

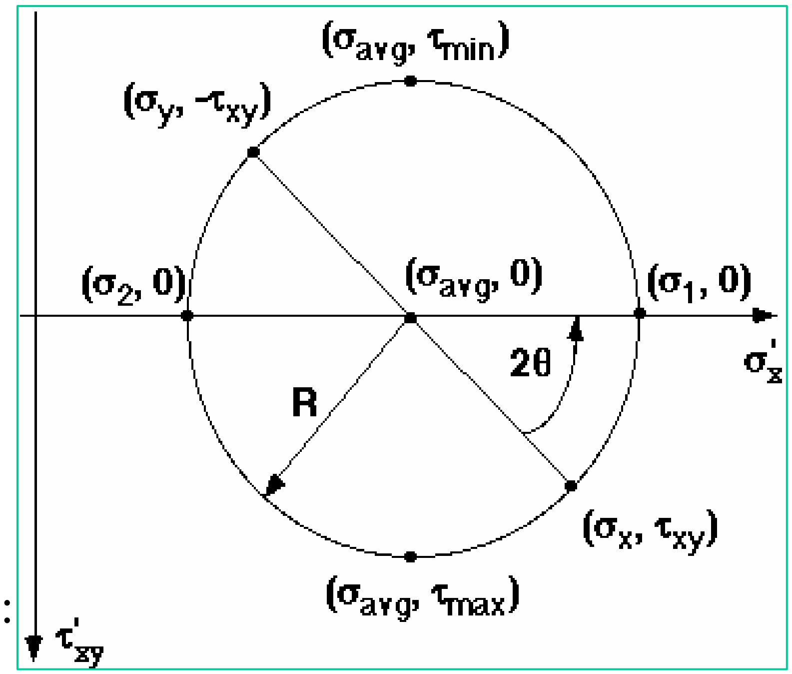
- The above equation is for a circle of radius R and Center σ_{avg}

Mohr's Circle for Plane Stress

- Mohr's circle equation

$$\left(\sigma'_x - \sigma_{avg}\right)^2 + \left(\tau'_{xy}\right)^2 = R^2$$

- Equation of a circle in the σ'_x-τ'_{xy} plane centered at |(σ_{avg}, 0) and radius R



* Every plane becomes a point on the circle.

* The intersection with the σ'_x axis defines the principal stresses.

$$\sigma_1 = \sigma_{avg} + R$$

$$\sigma_2 = \sigma_{avg} - R$$

* The bottom and top center positions correspond to:

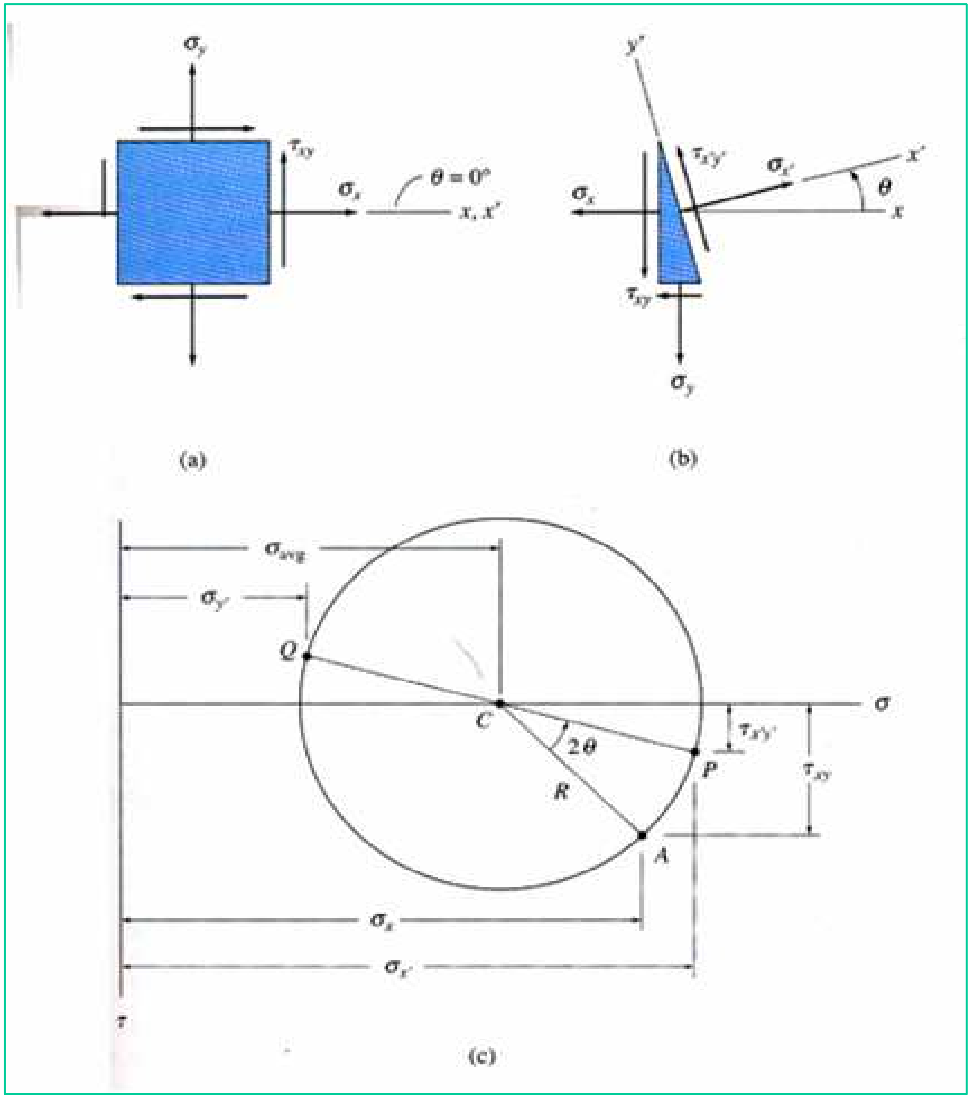
$$\tau_{max} = R \text{ and } \tau_{min} = -R$$

Procedures to Construct Mohr's Circle

With σ_x, σ_y and τ_{xy} known, the procedure for constructing Mohr's circle is as follows;

- 1) Draw a set of coordinate axes with σ as abscissa (positive to the right) and τ as ordinate (positive upward)
- 2) Locate the center C of the circle at the point having coordinates(σ_{aver}, 0)
- 3) Locate point A, representing the stress conditions on the face A (σ_x, - τ_{xy})
- 4) Locate point B, representing the stress conditions on the face B (σ_y, τ_{xy})
- 5) Draw a line from point A to point B. This line is a diameter of the circle and passes through the center C
- 6) Using point C as the center, draw Mohr's circle through points A and B.
- 7) On the circle, we measure an angle 2θ clockwise from radius CA. The angle 2θ locates point D.
- Point D on the circle represents the stresses on the face D of the element.
- Note that an angle 2θ on Mohr's circle corresponds to an angle θ on a stress element.

Procedures to Construct Mohr's Circle



Plane Strain

- Plane strain is defined by the strain state (ε_x ε_y γ_{xy}) ; it is the limiting condition in the center plane of a very thick specimen.

- Consider a rectangular element of material, OABC, in the xy-plane shown in Figure; it is required to find the normal and shearing strains in the direction of the diagonal OB, when the normal and shearing strains in the directions Ox, Oy are given.

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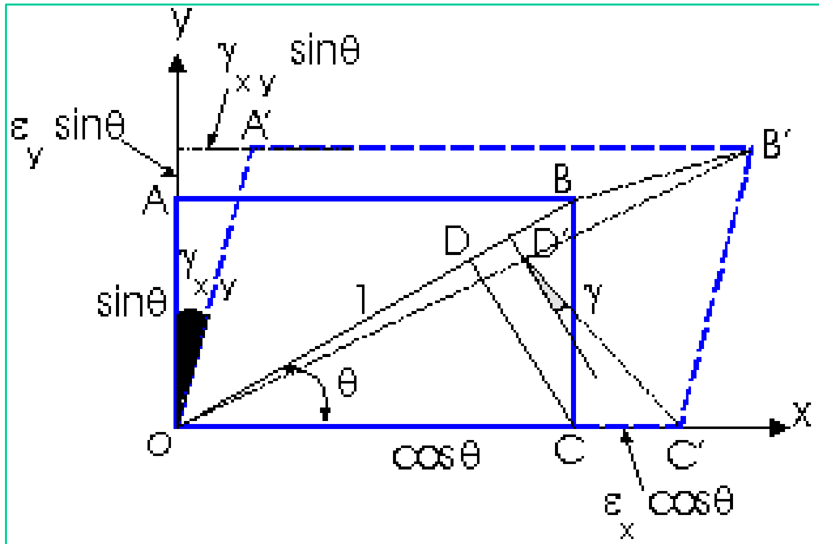


' Strains in an inclined direction; strains in the directions Ox, Oy and defined by ε_x, ε_y, γ_{xy}, lead to strains ε, γ along the inclined direction *OB*

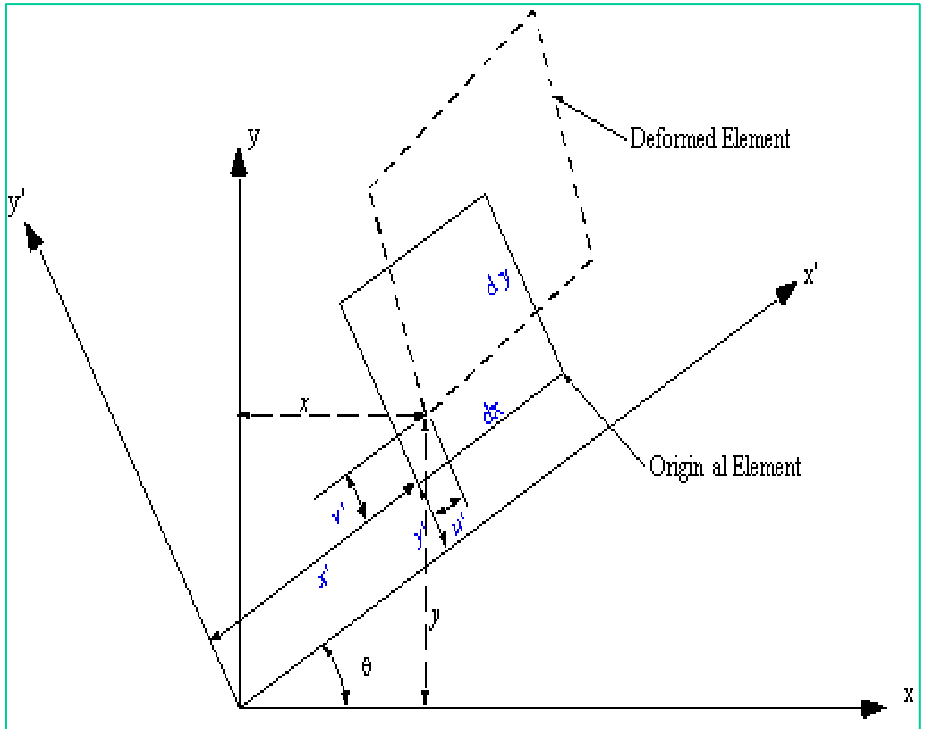
Strain Transformation

- Assume that strain transformation is desired from an *xy* coordinate system to an *xy′* set of axes, where the latter is rotated counterclockwise (+θ) from the *xy* system.
- The transformation equations for plane strain are

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$



Element of size dx by dy at angle θ before and after the application of biaxial stresses, showing its deformation

Principal Strains

- For isotropic materials only, principal strains (with no shear strain) occur along the principal axes for stress.
- In plane strain the principal strains ε_1 and ε_2 are expressed as
- The angular position θ_p of the principal axes (measured positive counterclockwise) with respect to the given *xy* system is determined from

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} +/- \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 - \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

Maximum Shear Strain

- Like in the case of stress, the maximum in-plane shear strain is:

$$\frac{\gamma_{x'y'max}}{2} = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

which occurs along axes at 45° from the principal axes, determined from θ_s

$$\tan 2\theta = -\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}$$

- The corresponding average normal strain is

$$\varepsilon_{avr} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

Mohr's Circle of Strain

- The direct and shearing strains in an inclined direction are given by relations which are similar to the Equations for the direct and shearing stresses on an inclined plane.
- This suggests that the strains in any direction can be represented graphically in a similar way to the stress system.

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\Rightarrow \sigma'_x + \sigma'_y = \sigma_x + \sigma_y$$

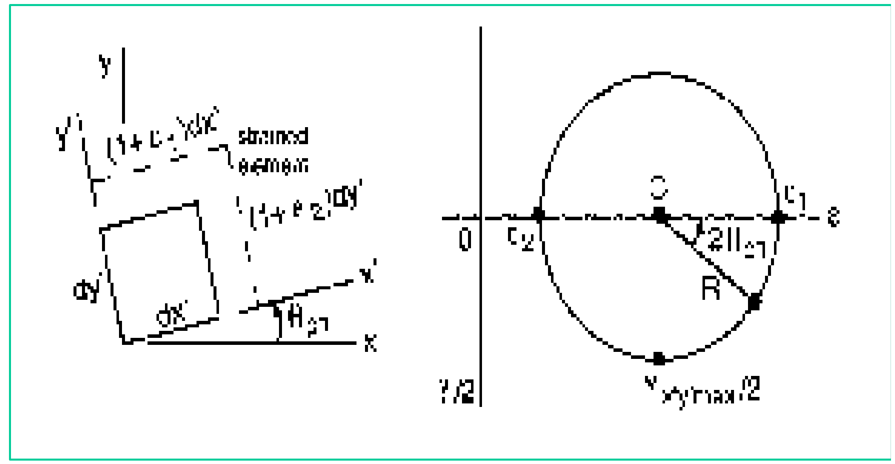
$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Mohr's Circle of Strain

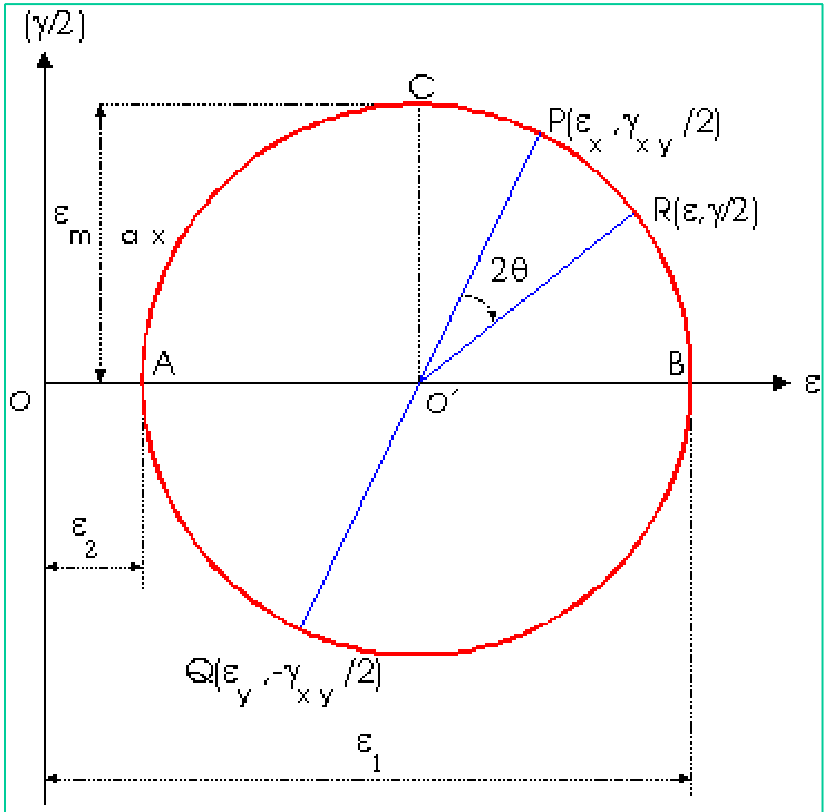
- As in the case of stress, there is a graphical overview by Mohr's circle of the directional dependence of the normal and shear strain components at a point in a material. This circle has a center C at $\varepsilon_{ave} = (\varepsilon x + \varepsilon y)/2$ which is always on the ε axis, but is shifting left and right in a dynamic loading situation. The radius R of the circle is



$$R = \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Mohr’s Circle of Strain

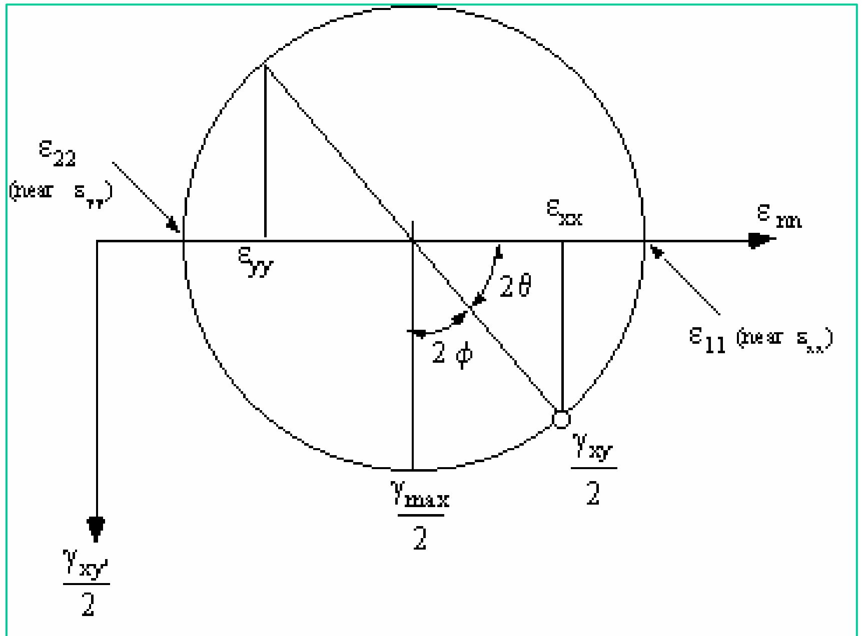
- For given values of ε_x, ε_y and γ_{xy} it is constructed in the following way:
- Two mutually perpendicular axes, ε and $\gamma/2$, are set up
- The points (ε_x, $\gamma_{xy}/2$) and (ε_y, $- \gamma_{xy}/2$) are located; the line joining these points is a diameter of the circle of strain.
- The values of ε and $\gamma/2$ in an inclined direction making an angle θ with Ox are given by the points on the circle at the ends of a diameter making an angle 2θ with PQ; the angle 2θ is measured clockwise.



Mohr's circle of strain; the diagram is similar to the circle of stress, except that γ/2 is plotted along the ordinates and not γ.

Mohr's Circle of Strain

- We note that the maximum and minimum values of ε, given by ε_1 and ε_2 occur when γ/2 is zero; ε_1, ε_2 are called principal strains, and occur for directions in which there is no shearing strain.

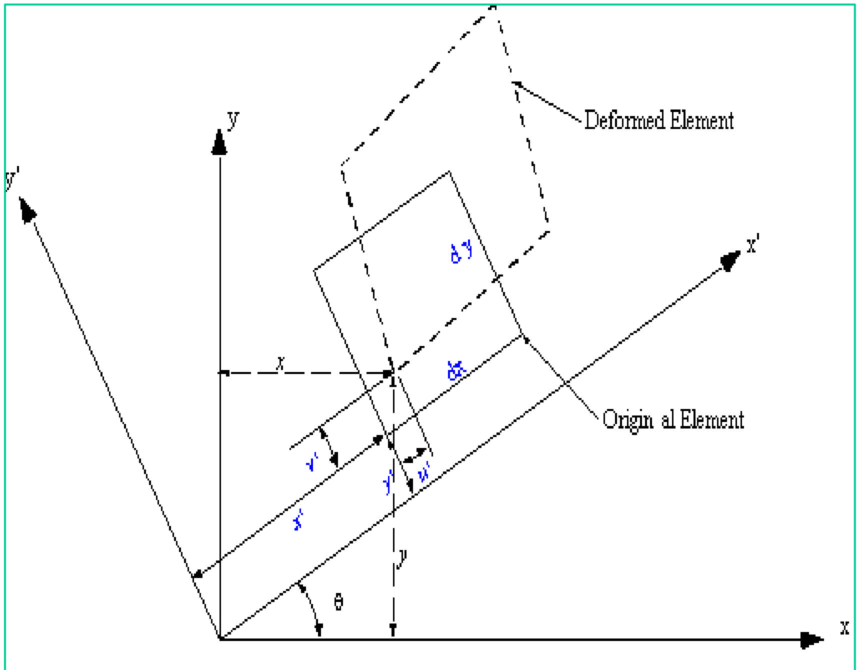


Mohr Circle for Strain

Strain Rosette

- Define the terms ε_{xx} ε_{yy} γ_{xy} as the strains of an element of size (dx*dy) at an angle θ with respect to the horizontal axis.
- Then the equations which defines these strains are:

$$\varepsilon_{xx'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$



Element of size dx by dy at angle θ before and after the application of biaxial stresses, showing its deformation

- If the strain at any angle could be measured,the equation above can then be used to determine the direct and shear strains in the structure about the x & y axes.

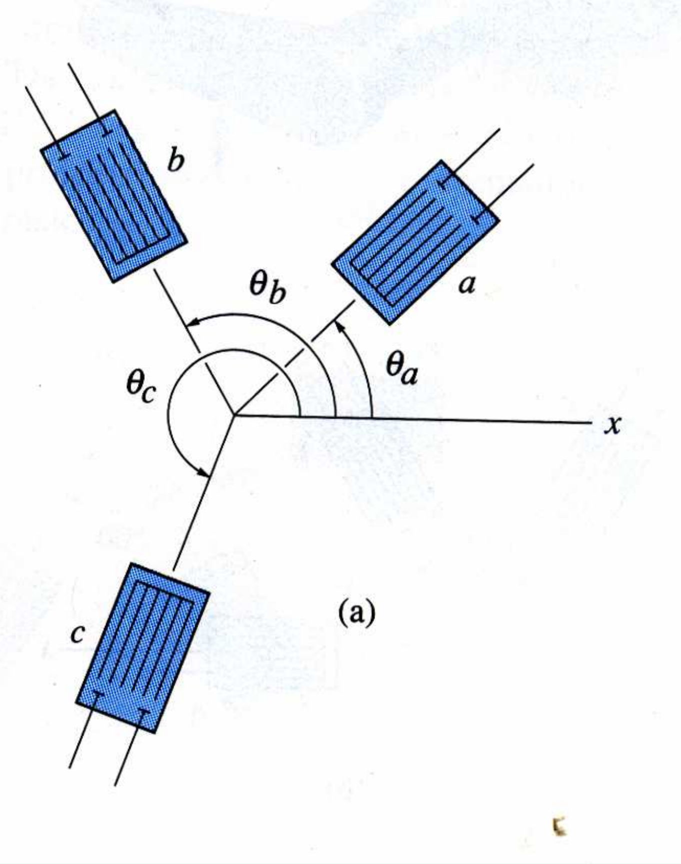
- These measurements are done using a Strain Gauge Rosette.

Strain Rosette

- A normal arrangement is to have three strain gauges oriented at three different angles w.r.t the horizontal axis of the structure, like this:
- Because we have three unknowns terms and you want to find, ε_{xx} ε_{yy} γ_{xy} ,use equation

$$\varepsilon_{xx'} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\cos\theta\sin\theta$$

three times, once for each angle. Then solve for the three strain



$$\varepsilon_{\theta_a} = \varepsilon_x \cos^2\theta_a + \varepsilon_y \sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a$$

$$\varepsilon_{\theta_b} = \varepsilon_x \cos^2\theta_b + \varepsilon_y \sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b$$

$$\varepsilon_{\theta_c} = \varepsilon_x \cos^2\theta_c + \varepsilon_y \sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c$$

Strain Rosette

- Strain rosettes are often arranged in 45 or 60 patterns, such that the solutions for the unknowns will be as follows:

For the 45^0 *Rosette*: $(\theta_a = 0^0, \theta_b = 45^0, \theta_c = 90^0)$

$$\varepsilon_x = \varepsilon_a$$

$$\varepsilon_y = \varepsilon_c$$

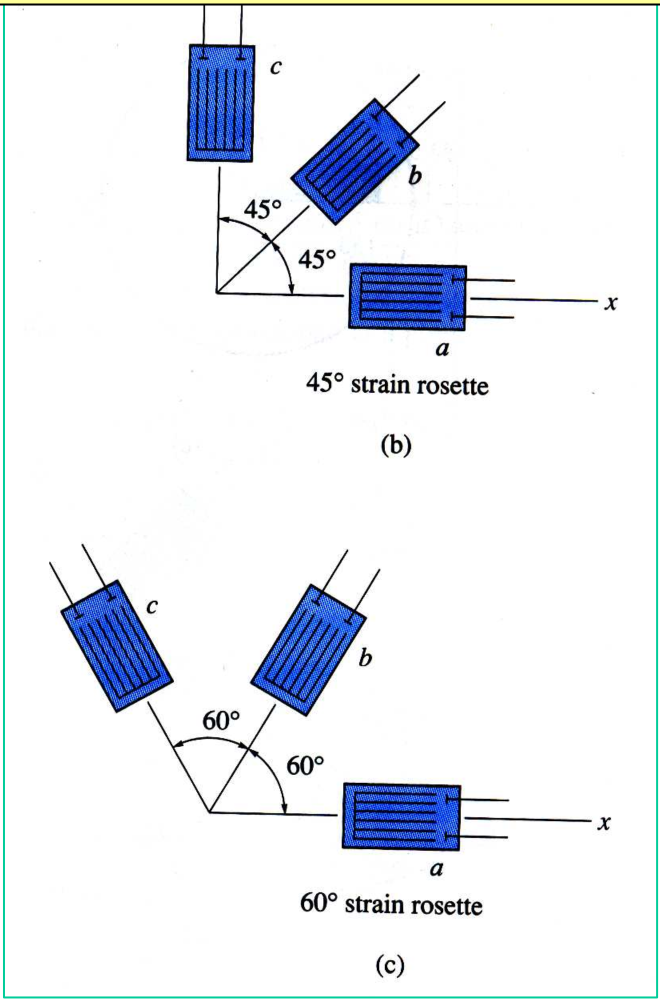
$$\gamma_{xy} = 2\varepsilon_b - (\varepsilon_a + \varepsilon_c)$$

For the 60^0 *Rosette*: $(\theta_a = 0^0, \theta_b = 60^0, \theta_c = 120^0)$

$$\varepsilon_x = \varepsilon_a$$

$$\varepsilon_y = \frac{1}{3}\left(2\varepsilon_b + 2\varepsilon_c - \varepsilon_a\right)$$

$$\gamma_{xy} = \frac{2}{\sqrt{3}}\left(\varepsilon_b - \varepsilon_c\right)$$



Material- Property Relationships

- Generalized Hooks Law: Once we have the strains use the relationships between stress and strain to find the stresses: (for isotropic material)

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_y + \sigma_x)\right)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

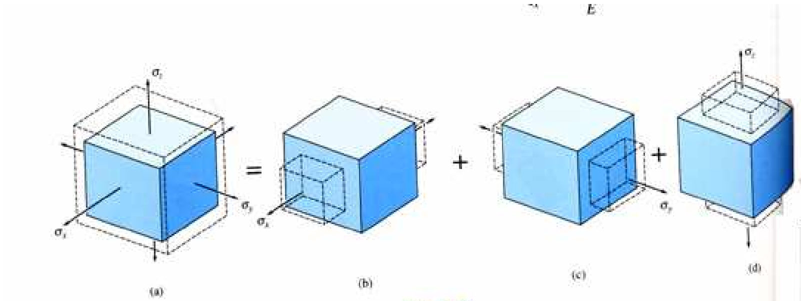
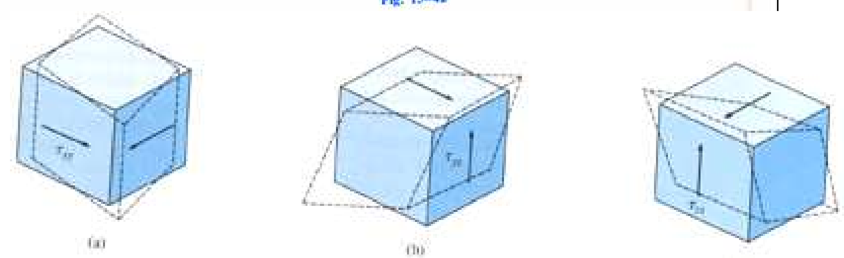


Fig. 15-42



Where

E: Young's modulus,

ν: Poisson's ratio

G: Shear modulus

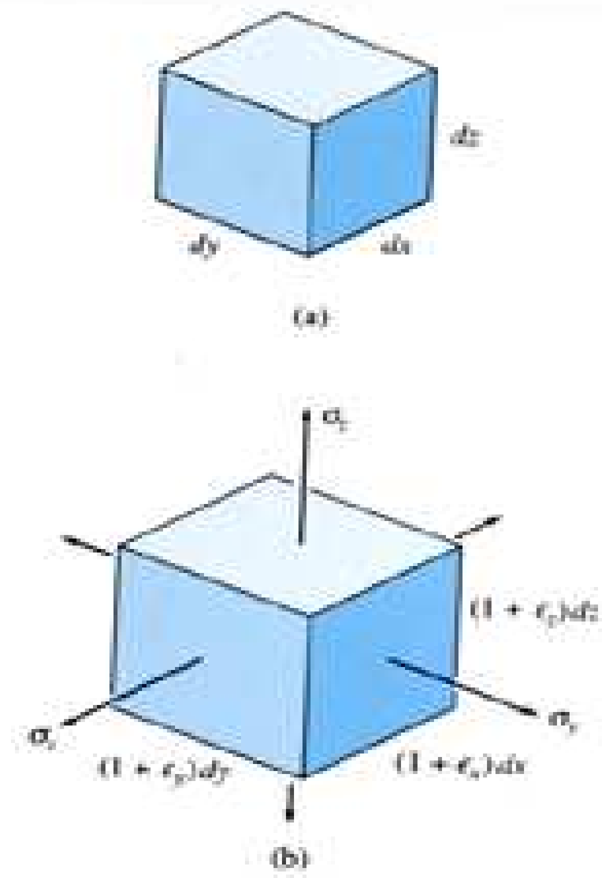
$$G = \frac{E}{2(1+\nu)}$$

Dilatation

- Under the application of normal stresses, the volume of the material will change.

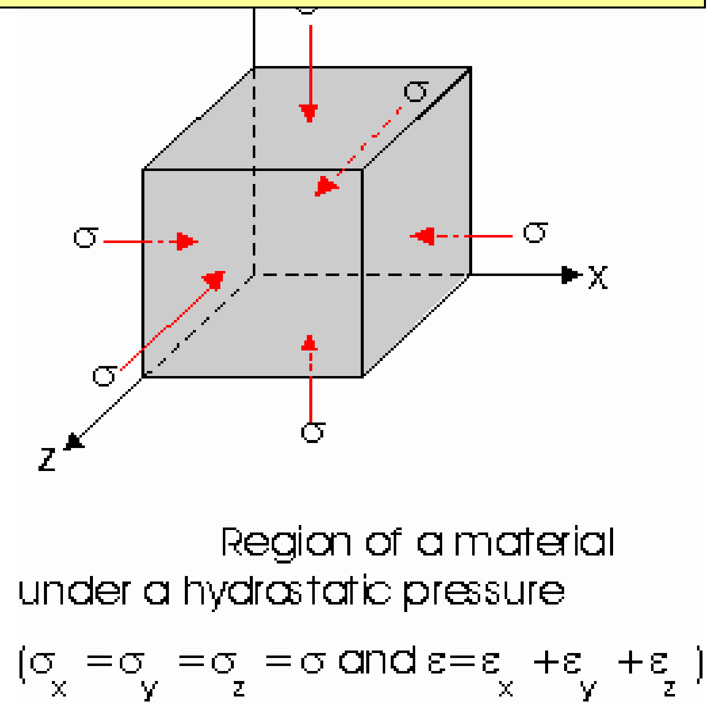
- The change in volume per unit volume(dV/V) is called the dilatation: *e*

$$e = \frac{dV}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$



Bulk Modulus

- A material under the action of equal compressive stresses s in three mutually perpendicular directions, is subjected to a hydrostatic pressure, *p*. The term hydrostatic is used because the material is subjected to the same stresses as would occur if it were immersed in a fluid at a considerable depth.



- The ratio between the hydrostatic pressure and the dilatation is called the Bulk modulus : k

$$k = \frac{p}{e} = \frac{E}{3(1-2\nu)}$$